

Engineering Notes

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Finite Element Model Updating Using Bayesian Framework and Modal Properties

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I. Introduction

FINITE element (FE) models are widely used to predict the dynamic characteristics of aerospace structures. These models often give results that differ from measured results and therefore need to be updated to match measured results. Some of the updating techniques that have been proposed to date use time, modal, frequency, and time–frequency domain data.^{1,2} In this Note, we use the modal domain data to update the FE model. A literature review on FE updating¹ reveals that the updating problem has been framed mainly in the maximum-likelihood framework. Even though this framework has been applied successfully in industry, it has the following shortcomings: it does not offer the user confidence intervals for solutions it gives; there is no philosophical explanation of the regularization terms that are used to control the complexity of the updated model; and it cannot handle the inherent ill-conditioning and nonuniqueness of the FE updating problem. In this Note the Bayesian framework is adopted to address the shortcomings explained above. The Bayesian framework has been found to offer several advantages over maximum-likelihood methods in areas closely mirroring FE updating.^{3–5} This Note seeks to address the following issues: 1) how prior information is incorporated into the FE model updating problem and 2) how to apply the Bayesian framework to update FE models to match experimentally measured modal properties (i.e., natural frequencies and mode shapes) to modal properties calculated from the FE model of a beam. In this Note, Markov chain Monte Carlo (MCMC) simulation³ is used to sample the probability of the updating parameters in light of the measured modal properties. This probability is known as the posterior probability. The Metropolis algorithm (see Ref. 6) is used as an acceptance criterion when the posterior probability is sampled.

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II. Mathematical Foundation

A. Dynamics

All elastic structures may be described in terms of their distributed mass, damping, and stiffness matrices. If damping terms are neglected, the dynamic equation may be written in the modal domain (natural frequencies and mode shapes) for the i th mode as follows⁷:

$$(-\omega_i^2[M] + [K])\{\phi\}_i = \{\varepsilon\}_i \quad (1)$$

Here $[M]$ is the mass matrix, $[K]$ is the stiffness matrix ω_i is the i th natural frequency, $\{\phi\}_i$ is the i th mode shape vector, and $\{\varepsilon\}_i$ is the i th error vector. The error vector $\{\varepsilon\}_i$ is equal to $\{0\}$ if the system matrices $[M]$ and $[K]$ correspond to the modal properties. If the system matrices, which are usually obtained from the FE model, do not match the measured modal properties ω_i and $\{\phi\}_i$, then $\{\varepsilon\}_i$ is a nonzero vector. In the maximum-likelihood method the Euclidean norm of $\{\varepsilon\}_i$ is minimized in order to match the system matrices to measured modal properties. Another problem that is encountered in many practical situations is that the dimension of mode shapes does not match the dimension of system matrices. This is because measured modal coordinates are fewer than FE modal coordinates. To ensure compatibility between system matrices and mode shape vectors, the dimension of system matrices is reduced using a technique called the Guyan reduction method⁸ to match the dimension of system matrices to the dimension of measured mode shape coordinates.

B. Bayesian Method

In this Note the Bayesian method is introduced to solve the FE updating problem based on modal properties. The fundamental rule that governs the Bayesian approach is written as follows³:

$$P(\{E\}|[D]) = \frac{P([D]|\{E\})P(\{E\})}{P([D])} \quad (2)$$

Here $\{E\}$ is a vector of updating parameters, $P(\{E\})$ is the probability distribution function of updating parameters in the absence of any data, known as the prior distribution, and $[D]$ is a matrix containing natural frequencies ω_i and mode shapes $\{\phi\}_i$. It must be noted that the mass $[M]$ and stiffness $[K]$ matrices are functions of updating parameters $\{E\}$. The quantity $P(\{E\}|[D])$ is the posterior distribution function after a set of data has been seen, $P([D]|\{E\})$ is the likelihood distribution function, and $P([D])$ is the normalization factor.

1. Likelihood Distribution Function

There are many areas where the likelihood distribution function has been applied, and these include neural networks.³ In a neural-network context, the likelihood distribution function is defined as the normalized exponent of the error function. In this Note the likelihood distribution function $P([D]|\{E\})$ is defined as the sum of squares of elements of the error vector shown in Eq. (1) and can be written in the same way as in neural networks, as follows³:

$$\begin{aligned} P([D]|\{E\}) &= \frac{1}{Z_D(\beta)} \exp \left[-\beta \sum_j^F \sum_i^N (\varepsilon_{ij}^2) \right] \\ &= \frac{1}{Z_D(\beta)} \exp \left(-\beta \sum_j^F \sum_i^N [(-\omega_i^2[M] + [K])\{\phi\}_i]_j^2 \right) \end{aligned} \quad (3)$$

Here β is the coefficient of the measured modal property data contribution to the error and is set to 1 through trial and error, and ε_{ij} is the error matrix with subscript i representing the i th modal properties and j representing the j th measurement position. The superscript F is the number of measured mode-shape coordinates, N is the number of measured modes, and Z_D is

$$Z_D(\beta) = \int \exp\left(-\beta \sum_j^F \sum_i^N ([(-\omega_i^2[M] + [K])\{\phi\}_i]_j)^2\right) d[D] \quad (4)$$

It should be noted that in Eq. (3) the error ε_{ij} is a matrix, as opposed to a vector as is the case in Eq. (1). This is because it takes into account of all modal coordinates.

2. Prior Distribution Function of Parameters to Be Updated

The prior distribution function consists of the information that is known about the problem. In FE updating it is generally accepted that FE updating is usually valid if the model is close to the true model. In this Note, it is known that not all parameters to be updated have the same level of modeling error. This means that some parameters are to be updated more intensely than others. For example, parameters next to joints should be updated more intensely than those with smooth surface areas that are far from joints. In this Note the prior distribution function for parameters to be updated may be written using the Gaussian assumption as follows³:

$$P(\{E\}) = \frac{1}{Z_E(\alpha)} \exp\left(-\sum_i^Q \frac{\alpha_i}{2} \|\{E\}\|^2\right) \quad (5)$$

Here Q is the number of groups of parameters to be updated and α_i is the coefficient of the prior distribution function for the i th group of updating parameters. The prior distribution function in Eq. (5) ensures that large updating of parameters is less likely than small adjustments of updating parameters. The Gaussian prior has been successfully used³ to identify a large number of weights in neural networks, and therefore it is assumed that it should be successful in identifying a small number of updating parameters in this Note. The higher the α_i the lower is the degree of updating of the i th group of parameters, and $\|\cdot\|$ is the Euclidean norm of \cdot . In Eq. (5), if α_i is constant for all the updating parameters, then the updated parameters will be of the same order of magnitudes. Equation (5) may be viewed as a regularization parameter.⁹ In Eq. (5), Gaussian priors are conveniently chosen because many natural processes tend to have Gaussian distributions. In the Bayesian framework the regularization method is viewed as a mechanism of incorporating prior information, whereas in the maximum-likelihood method it is viewed as a mathematical convenience. The function $Z_E(\alpha)$ is a normalization factor given by Ref. 3:

$$Z_E(\alpha) = \int \exp\left(-\sum_i^Q \frac{\alpha_i}{2} \|\{E\}\|^2\right) d\alpha_i \quad (6)$$

3. Posterior Distribution Function of Weight Vector

The distribution of the weights $P(\{E\}|[D])$ after the data have been seen is calculated by substituting Eqs. (3) and (5) into Eq. (2) to give

$$P(\{E\}|[D]) = \frac{1}{Z_s(\alpha, \beta)} \exp\left(-\beta \sum_j^F \sum_i^N ([(-\omega_i^2[M] + [K])\{\phi\}_i]_j)^2 - \sum_i^Q \frac{\alpha_i}{2} \|\{E\}\|^2\right) \quad (7)$$

where

$$Z_s(\alpha, \beta) = P([D]) = \int \exp\left(-\beta \sum_j^F \sum_i^N ([(-\omega_i^2[M] + [K])\{\phi\}_i]_j)^2 - \sum_i^Q \frac{\alpha_i}{2} \|\{E\}\|^2\right) d\{E\} \quad (8)$$

In Eq. (7), the optimal weight vector corresponds to the maximum of the posterior distribution function, which is the solution as obtained from a maximum-likelihood approach. This implies that the Bayesian method at least gives the solution that is given by the maximum-likelihood method but in addition gives probability distributions.

C. Markov Chain Monte Carlo Method

The application of the Bayesian approach to FE model updating using a Monte Carlo approach results in a set of updated parameter vectors $\{E\}_i$ that are statistical rather than deterministic. As a result, FE model updating will give distributions of the predicted modal properties, and from these distributions, averages and variances of modal properties may be constructed. Following the rules of probability theory, the distribution of the vector $\{Y\}$, representing measured modal properties, may be written in the following form:

$$P(\{Y\}|[D]) = \int P(\{y\}|[E])P(\{E\}|[D]) d\{E\} \quad (9)$$

Equation (9) depends on Eq. (7) and is difficult to solve analytically due to the relatively high dimension of the updating parameter vector. As a result, a Markov chain Monte Carlo (MCMC) method is employed to determine the distribution of updating parameters and subsequently of predicted modal properties. The integral in Eq. (9) is solved, using the Metropolis algorithm,⁶ by generating a sequence of vectors $\{E\}_1, \{E\}_2, \dots$ that form a Markov chain with a stationary distribution $P([D]|\{E\})$. The integral in Eq. (9) may thus be approximated as follows:

$$\{\bar{y}\} \cong \frac{1}{L} \sum_{i=1}^{R+J-1} G(\{E\}_i) \quad (10)$$

Here G is a finite element model that takes the vector $\{E\}_i$ and predicts the average output, $\{\bar{y}\}$ is the vector containing the modal properties, R is the number of initial states that are discarded in the hope of reaching a stationary distribution described by Eq. (7), and L is the number of retained states. Several methods have been proposed to simulate the distribution in Eq. (7), such as Gibbs sampling,¹⁰ the Metropolis algorithm,⁶ and a hybrid Monte Carlo method.¹¹ Hybrid Monte Carlo, which has been shown to be the most efficient of the Monte Carlo methods thus far, is not used in this Note because it requires gradient information, which is not available in exact form in the FE updating problem. As a result, the MCMC method is used to identify the posterior distribution function of the updating parameters. In this Note the MCMC method is implemented by sampling a stochastic process consisting of random variables $\{\{E\}_1, \{E\}_2, \dots, \{E\}_n\}$ through introduction of random changes to the updating parameter vector $\{E\}$ and either accepting or rejecting the sample according to the Metropolis algorithm.⁶ Metropolis criteria can be written as follows:

if $P_{\text{new}}(\{E\}|[D]) > P_{\text{old}}(\{E\}|[D])$ accept state $\{E\}_{\text{new}}$

else accept $\{E\}_{\text{new}}$ with probability $\frac{P_{\text{new}}(\{E\}|[D])}{P_{\text{old}}(\{E\}|[D])}$ (11)

In this Note we view this procedure as a way of generating a Markov chain with transition from one state to another, conducted using the criterion in Eq. (11).

III. Example: Experimentally Measured Beam

To test the proposed procedure a freely suspended aluminum beam is used. The beam, which is shown in Fig. 1, has the following dimensions: length 1.0 m, width 25.4 mm, and thickness 13.4 mm. Acceleration measurements are taken at 13 equidistant positions and the beam is excited at a position located 420 mm from the end (see Fig. 1). Further details of this beam are found in Ref. 12. The FE model with 12 elements is constructed using a modulus of elasticity of $800 \times 10^8 \text{ N m}^{-2}$ and a density of 2700 kg m^{-3} . Using conventional signal-processing analysis,⁷ the measured data

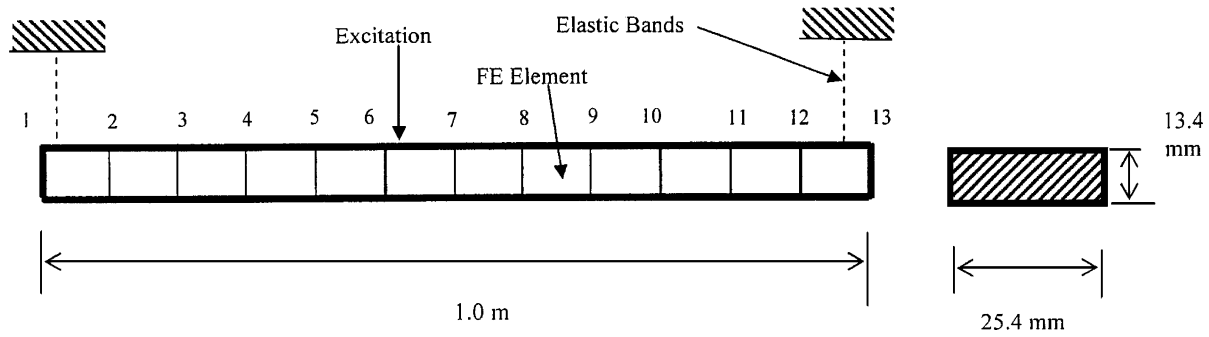


Fig. 1 Diagram showing a beam: its cross-sectional area, elastic bands used for suspension, and positions where acceleration measurements were taken.

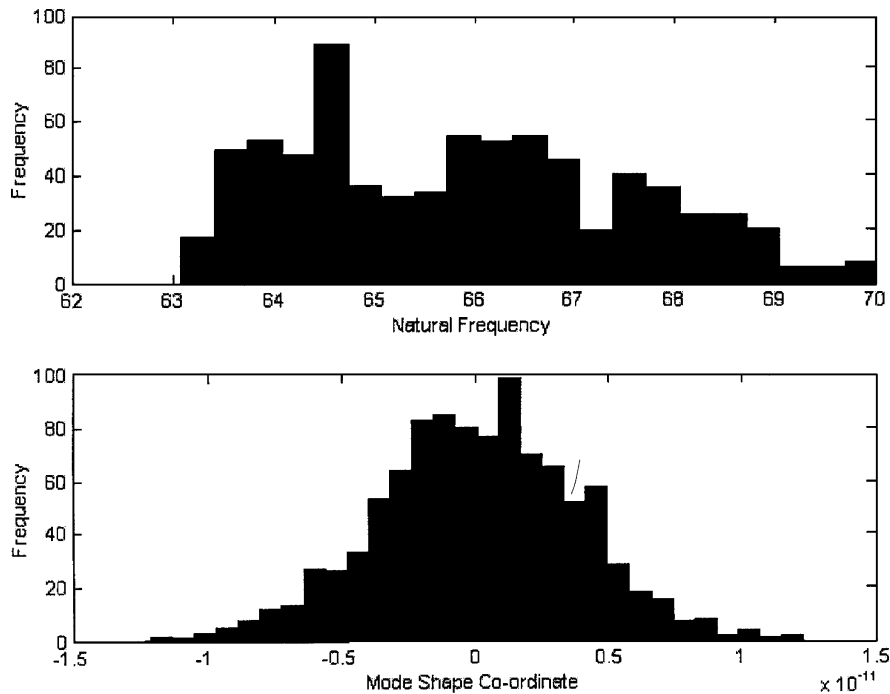


Fig. 2 Sample distributions of the natural frequencies and mode shape coordinate of the updated finite element model prediction.

are transformed into frequency response functions (FRFs) and from the FRFs, natural frequencies and mode shapes are extracted using modal extraction techniques.⁷ Using the extracted natural frequencies and mode shapes the FE model is updated using the Bayesian framework. When the Bayesian framework is applied, Eq. (7) is used, prior information is divided into four parts, and each part has its own coefficient of prior distribution (α_1 , α_2 , α_3 , and α_4). These coefficients are also shown in Eq. (7) by setting Q equal to 4. The coefficient α_1 is associated with the density of the beam, is known to be uniform for all elements, and is also known to be fairly accurate. The coefficient α_1 is set to 10 to ensure that the density of the beam is not updated significantly. The coefficient α_2 is associated with the moduli of elasticity of all elements. All elements are known to have uniform modulus of elasticity, which is known fairly accurately. The coefficient α_2 is set to 10 to ensure that the modulus of elasticity is not updated significantly. The coefficient α_3 is associated with the cross-sectional areas of elements 1–4 and 7–12, which are known fairly accurately. The coefficient α_3 is set to 10 to ensure that the cross-sectional areas of these elements are not updated significantly. The coefficient α_4 is associated with the cross-sectional areas of elements 5 and 6, which are not known accurately because they enclose the area that was drilled to mount the excitation device. The coefficient α_4 is set to 0.1 to ensure that the cross-sectional areas of these elements are updated significantly. The MCMC method is implemented by employing the Metropolis

Table 1 Measured natural frequencies, frequencies calculated from initial and updated finite element models and associated standard deviations

Mode number	Experiment, Hz	Initial FEM, Hz	Average updated, Hz	Standard deviation, Hz
1	64	70	67	2.8
2	184	193	183	7.6
3	349	379	360	16.1
4	599	628	590	28.7
5	898	942	893	76.4

acceptance criterion [see Eq. (11)] and 1000 samples are retained to form a posterior distribution function indicated by Eq. (7).

IV. Discussion

When natural frequencies from the updated FE model are compared to those calculated from the initial FE model as well as those from the measured natural frequency data, the results in Table 1 are obtained. Table 1 also shows standard deviations of the distributions obtained through the use of the MCMC method to sample the distribution in Eq. (7). The updated natural frequencies are calculated using Eq. (10). This table shows that for all the modes the updated model is more accurate than the initial model. Furthermore, it is

Table 2 Modal assurance criterion between the measured mode shapes and finite element model-calculated mode shapes as well as associated standard deviations

Mode number	MAC experiment/initial	Average MAC experiment/updated	Standard deviation
1	0.9961	0.9992	0.0011
2	0.9895	0.9974	0.0019
3	0.9799	0.9958	0.0029
4	0.9703	0.9981	0.0011
5	0.9712	0.9949	0.0044

observed in Table 1 that the higher the mode, the higher the standard deviation, indicating that higher modes are less certain than lower modes. This is consistent with the knowledge that, in general, high-frequency modes are less certain than low-frequency modes. To compare analytical mode shapes to measured mode shapes, the modal assurance criterion (MAC) is used.¹³ The MAC is a criterion that represents how well two mode shapes are correlated. Two perfectly correlated mode shapes give an identity matrix. As a result, in this Note the diagonals of the MAC, whose elements are supposed to be equal to 1 for similar mode shapes, are used to assess the effectiveness of the proposed updating method. The diagonal of the MAC between mode shapes from experiment and from the updated FE models is shown in Table 2. This table shows that the updated FE model gives more accurate mode shapes than the initial FE model. Tables 1 and 2 show standard deviations, and these are used to construct error bars that measure confidence intervals of updated models. The results showing the distributions of the first natural frequency and mode shape coordinate are shown in Fig. 2. From these distributions error bars may be constructed for confidence intervals.

V. Conclusions

In this Note an updating procedure that uses a Bayesian framework and modal properties is implemented using a Markov chain Monte Carlo method. The method takes into account of prior information and has the advantage of giving distributions of pre-

dicted model properties. When the method is tested on experimental data it is found to significantly improve the accuracy of finite element models.

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